Math 5 – Trigonometry – fall '06 – Chapter 3 Test (2) Solutions

1. Compute and simplify the average rate of change of $f(x) = 2x^2 + 8x$ over the given interval. Remember that the average rate of change on the interval [a, b] is the slope of the secant line connecting [a, $f(a)$] with [b, $f(b)$].
   a. [0, 3]

      SOLN: $\dfrac{f(3) - f(0)}{3 - 0} = \dfrac{42 - 0}{3} = 14$

   b. [a, a+h]

      SOLN: $$\frac{f(a+h) - f(a)}{a+h-a} = \frac{2(a+h)^2 + 8(a+h) - 2a^2 - 8a}{h} = \frac{2a^2 + 4ah + 2h^2 - 2a^2 - 8a}{h}$$
      $$= \frac{4ah + 2h^2}{h} = 4a + 2h$$

2. Find the maximum value of the given function and state its range in interval notation.
   a. $f(x) = -2(x-3)^2 + 8$

      SOLN: The max is at (3, 8) and the range is $(-\infty, 8]$

   b. $f(x) = -2x^2 + 8x + 1$

      SOLN: $f(x) = -2x^2 + 8x + 1 = -2(x-2)^2 + 9$ has a max at (2, 9) and the range is $(-\infty, 9]$.

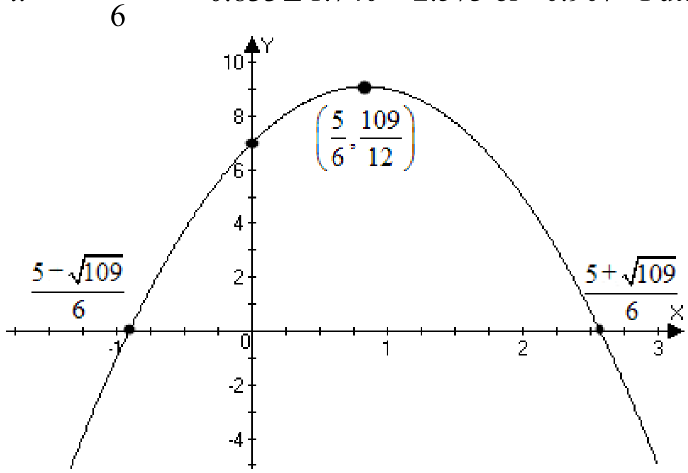
3. Consider the quadratic $f(x) = -3x^2 + 5x + 7$
   a. Express the quadratic function in standard form.

      SOLN: $f(x) = -3x^2 + 5x + 7 = -3\left(x - \dfrac{5}{6}\right)^2 + \dfrac{109}{12}$

   b. Sketch its graph.

      SOLN: The vertex is at $\left(\dfrac{5}{6}, \dfrac{109}{12}\right)$. The y-intercept is at (0,7) and the x-intercepts are where

      $x = \dfrac{5 \pm \sqrt{109}}{6} \approx 0.833 \pm 1.740 = 2.573$ or -0.907 Putting this together we have a nice graph:



   c. What transformations would be required to transform this function to $y = x^2$.

      SOLN: Shift 5/6 left, shift 109/12 down, reflect in the x-axis and compress vertically by 1/3.

4. Given the graph of $y = f(x)$ shown at right, graph

a. $y = 2f(x)$

SOLN: This is a vertical stretch by a factor 2, so the point at (1,2) so stretched to (1,4), the point at (3,3) is stretch to (3,6), the point at (-1,-2) is stretched down to (-1,-4) and the point at (-3,-3) is stretched to (-3,-6)

b. $y = f\left(\frac{x}{2}\right)$

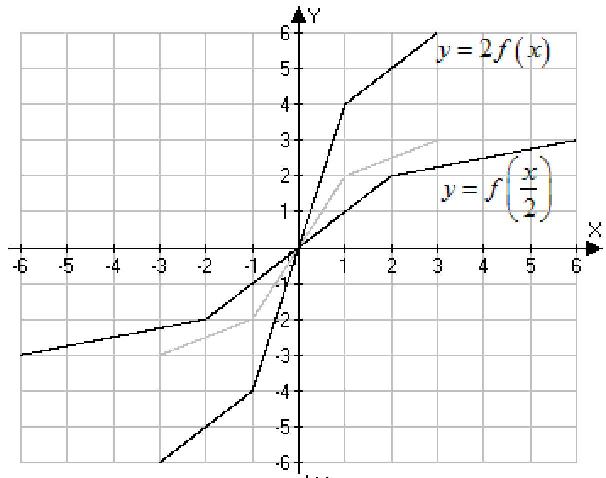
SOLN: This is a horizontal stretch by a factor of 2 so the point (1,2) is stretched to (2,2), the point at (3,3) is stretched to (6,3) and so on.

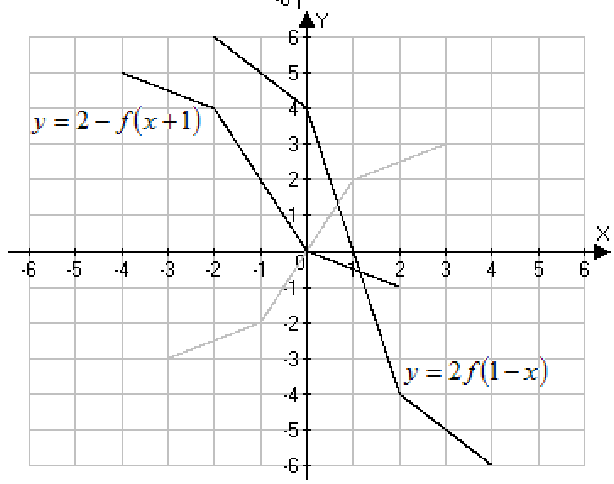
c. $y = 2f(1-x) = 2f(-(x-1))$

SOLN: First shift 1 left, then reflect in y-axis, then stretch vertically by a factor 2. Thus the point (1,2) goes to (0,2), stays there and is stretched to (0,4). Similarly (3,3) goes to (2,3) then (-2,3) then (-2,6), and the other points, similarly, to produce the graph labeled at right.

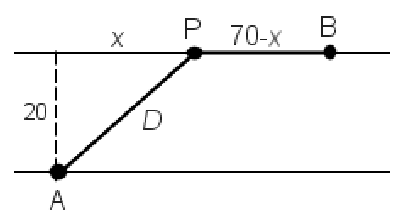
d. $y = 2 - f(x+1)$

This is a shift 1 to the left, followed by a reflection in the x axis and then a shift of 2 up. (1,2) goes to (0,2) then (0,-2) and then (0,0). The other points are similarly transformed to produce the graph labeled at right.





5. A mouse stands at point A on the bank of a straight canal, 20 feet wide. To reach point B, 70 feet down the canal on the opposite bank, it swims to a point P on the opposite bank and then crawls the remaining distance to B. The mouse swims at 5 feet per minute and crawls at 10 feet per minute. Model the total time of the mouse's trip from A to B as a function of where he lands on the opposite side.



SOLN: Let x = the distance down the shore the mouse lands on the other side (see diagram.) Then the remaining distance it needs to travel is $70 - x$. The distance D the mouse swims is the hypotenuse of a right triangle with legs of lengths 20 and x. Thus $D = \sqrt{20^2 + x^2}$. Using the formula $\text{time} = \frac{\text{distance}}{\text{speed}}$ we have the time swimming is $t_1 = \frac{\sqrt{x^2+400}}{5}$ minutes and the time crawling is $t_2 = \frac{70-x}{10}$, so the total time is $t = t_1 + t_2 = \frac{\sqrt{x^2+400}}{5} + \frac{70-x}{10}$

6. Consider $f(x) = x^2$

a. Write a formula for the function that results from shifting 2 units left, reflecting in the y-axis and then stretching horizontally by a factor 3, in that order.

SOLN: $f(x) = -\left(\frac{x}{3}+2\right)^2$

b. What transformations on $f(x)$, in order, would produce this formula: $y = 2-\left(\frac{x}{2}-1\right)^2$

SOLN: Shift 1 unit right, stretch horizontally by 2, reflect in the x-axis and shift up 2.

7. Suppose $f(x) = \sqrt{x-1}$ and $g(x) = \frac{1}{x-2}$.

a. Find the domain of $(f \circ g)(x)$
SOLN: First, x must be in the domain of g, so x is not 2. then the output of g must be greater than 1, so that f is real-valued. Thus the domain is the interval (2,3].

b. Find the domain of $(g \circ f)(x)$
SOLN: First, x must be in the domain of f, so $x \geq 1$. Then the output of f must not be 2, so x is not 5. Thus the domain is $[1,5) \cup (5,\infty)$.

8. Find a formula for the inverse function of $f(x) = (x+1)^3 - 3$ and sketch a graph for $f^{-1}(x)$ and $f(x)$ together showing the symmetry through the line $y = x$.

SOLN: $y = (x+1)^3 - 3 \Leftrightarrow x = -1+\sqrt[3]{y+3}$ so the inverse function is $f^{-1}(x) = -1+\sqrt[3]{x+3}$